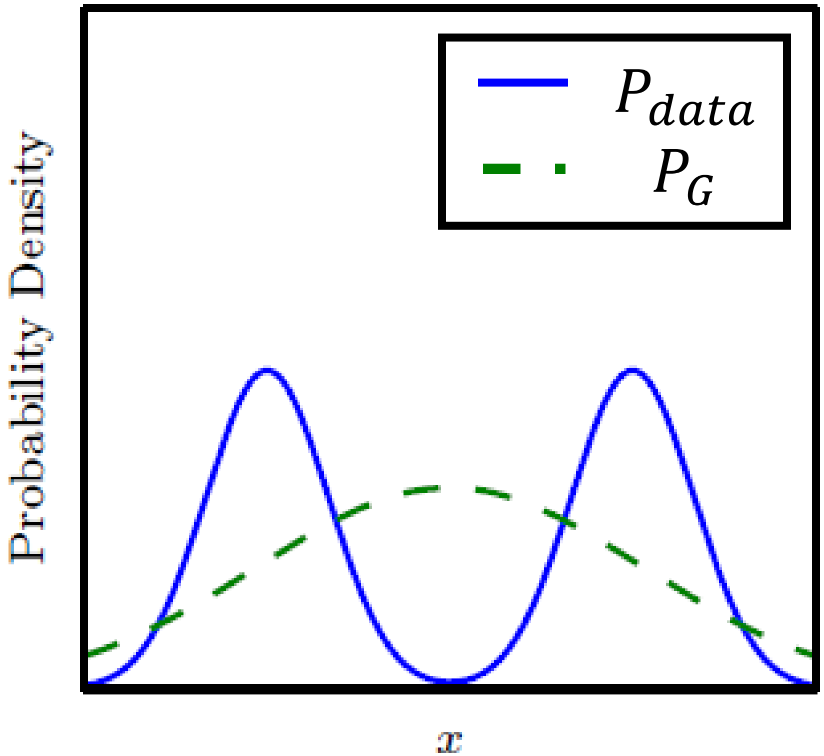
fGAN: General Framework of GAN

Modified from Ian Goodfellow's tutorial

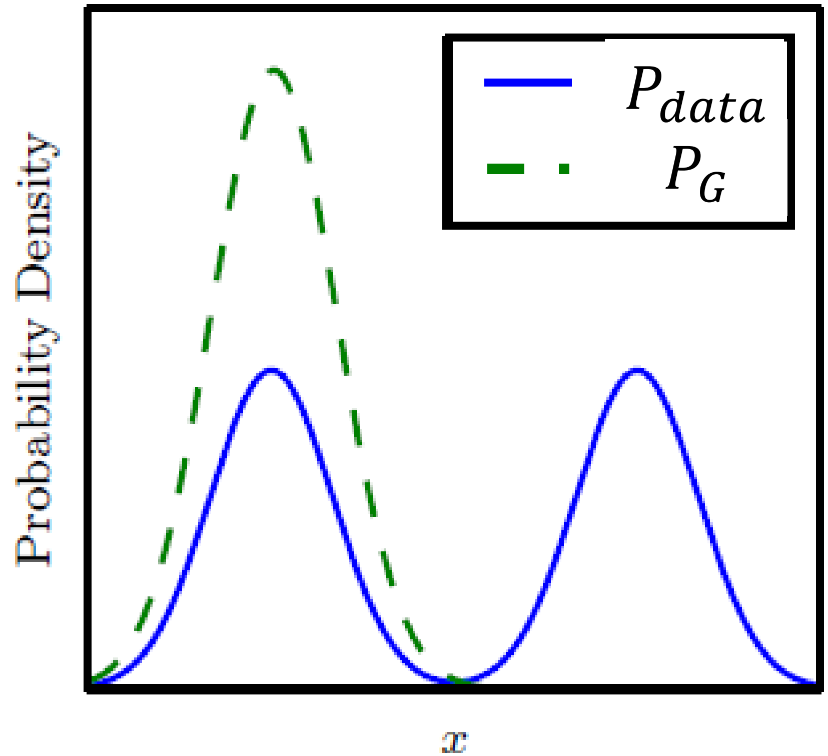
Flaw in Optimization?

$$KL = \int P_{data} log \frac{P_{data}}{P_G} dx$$

Reverse $KL = \int P_G log \frac{P_G}{P_{data}} dx$



Maximum likelihood
(minimize KL($P_{data}||P_G$))



Minimize KL($P_G||P_{data}$)
(reverse KL)

***f*-divergence**

P and Q are two distributions. $p(x)$ and q(x) are the probability of sampling x.

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

f is convex

f(1) = 0

$D_f(P||Q)$ evaluates the difference of P and Q

If $p(x) = q(x)$ for all x

smallest

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx = 0$$

$\frac{p(x)}{q(x)} = 1$, $f\left(\frac{p(x)}{q(x)}\right) = 0$

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

Because f is convex

$$\geq f\left(\int_x q(x) \frac{p(x)}{q(x)} dx\right)$$

$$= f(1) = 0$$

If P and Q are the same distributions, $D_f(P||Q)$ has the smallest value, which is 0

f-divergence

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

f is convex

f(1) = 0

$f(x) = x\log x$

KL

$$D_f(P||Q) = \int_x q(x) \frac{p(x)}{q(x)} \log\left(\frac{p(x)}{q(x)}\right) dx = \int_x p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

$f(x) = -\log x$

Reverse KL

$$D_f(P||Q) = \int_x q(x) \left(-\log\left(\frac{p(x)}{q(x)}\right)\right) dx = \int_x q(x) \log\left(\frac{q(x)}{p(x)}\right) dx$$

$f(x) = (x-1)^2$

Chi Square

$$D_f(P||Q) = \int_x q(x) \left(\frac{p(x)}{q(x)} - 1\right)^2 dx = \int_x \frac{\left(p(x) - q(x)\right)^2}{q(x)} dx$$

Fenchel Conjugate

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

f is convex, f(1) = 0

- Every convex function f has a conjugate function f*

$$f^*(t) = \max_{x \in dom(f)} \{xt - f(x)\}$$

$$f^*(t_1) = \max_{x \in dom(f)} \{x t_1 - f(x)\}$$

$$f^*(t_2) = \max_{x \in dom(f)} \{x t_2 - f(x)\}$$

$x_1 t_1 - f(x_1)$ $f^*(t_1)$

$x_2 t_1 - f(x_2)$

$x_3 t_1 - f(x_3)$

$x_3 t_2 - f(x_3)$ $f^*(t_2)$

$x_2 t_2 - f(x_2)$

$x_1 t_2 - f(x_1)$

t_1 t_2 t

Fenchel Conjugate

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

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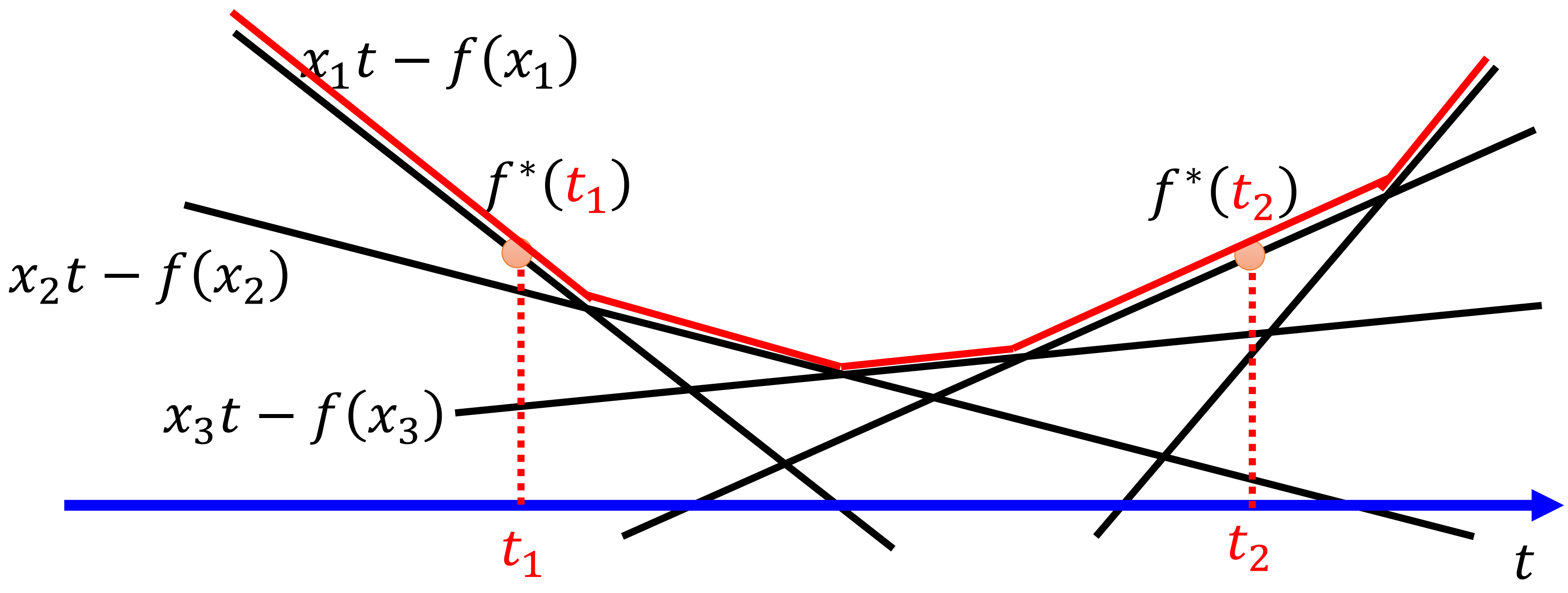


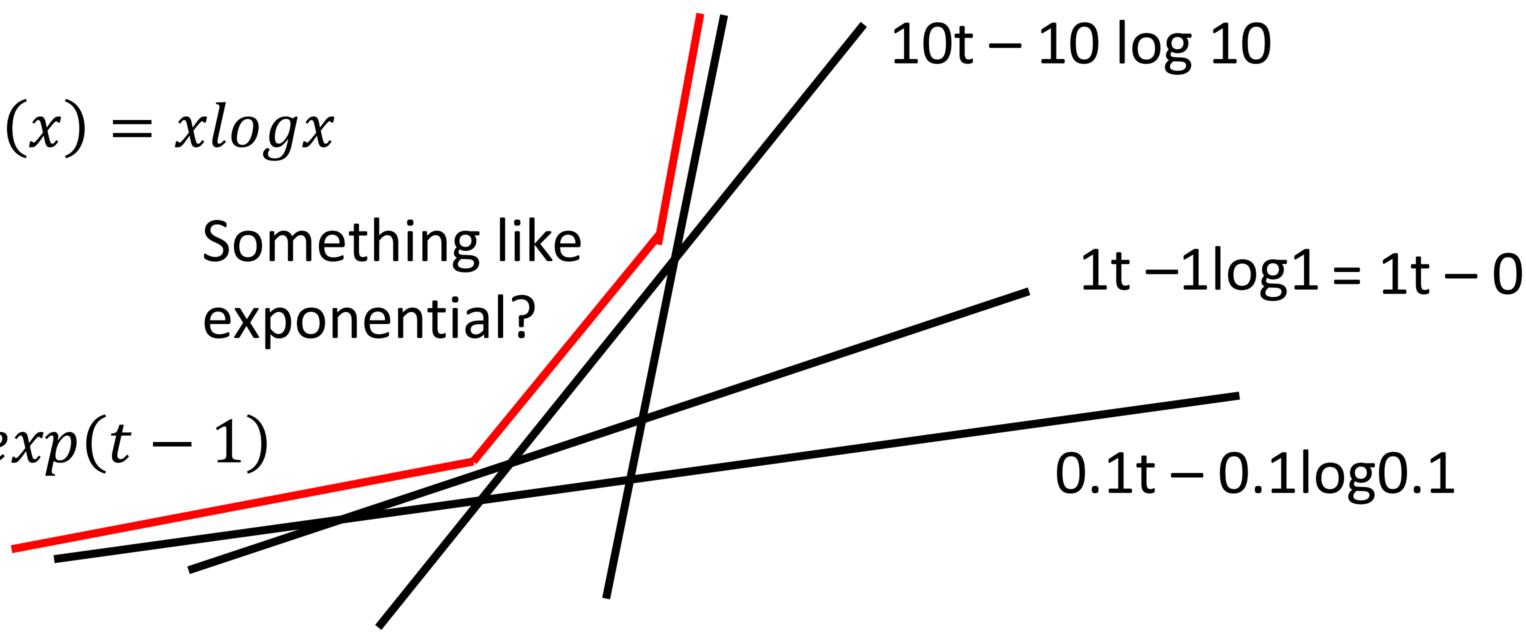
Fenchel Conjugate

- Every convex function f has a conjugate function f*

$$f^*(t) = \max_{x \in dom(f)} \{xt - f(x)\}$$

$$f(x) = xlogx$$

$$f^*(t) = exp(t-1)$$



Fenchel Conjugate

- Every convex function f has a conjugate function f*
- (f*)* = f

$$f^*(t) = \max_{x \in dom(f)} \{xt - f(x)\}$$

$$f(x) = xlogx \quad \longleftrightarrow \quad f^*(t) = exp(t-1)$$

$$f^*(t) = \max_{x \in dom(f)} \{xt - xlogx\}$$

$g(x) = xt - xlogx$ Given t, find x maximizing $g(x)$

$t - logx - 1 = 0$ $x = exp(t-1)$

$$f^*(t) = exp(t-1) \times t - exp(t-1) \times (t-1) = exp(t-1)$$

Connection with GAN

$$f^*(t) = \max_{x \in dom(f)} \{xt - f(x)\} \longleftrightarrow f(x) = \max_{t \in dom(f^*)} \{xt - f^*(t)\}$$

$$\frac{p(x)}{q(x)} \qquad \frac{p(x)}{q(x)}$$

$$D_f(P||Q) = \int_x q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

$$= \int_x q(x)\left(\max_{t \in dom(f^*)}\left\{\frac{p(x)}{q(x)} t - f^*(t)\right\}\right) dx$$

$$\approx \max_D \int_x p(x) D(x) dx - \int_x q(x) f^*(D(x)) dx$$

D is a function whose input is x, and output is t

$$D_f(P||Q) \geq \int_x q(x)\left(\frac{p(x)}{q(x)} D(x) - f^*(D(x))\right) dx$$

$$= \int_x p(x) D(x) dx - \int_x q(x) f^*(D(x)) dx$$

Connection with GAN

$$D_f(P||Q) \approx \max_{\mathrm{D}} \int_x p(x)D(x)dx - \int_x q(x)f^*\big(D(x)\big)dx$$

$$= \max_{\mathrm{D}} \{E_{x\sim P}[D(x)] - E_{x\sim Q}\big[f^*\big(D(x)\big)\big]\}$$

Samples from P Samples from Q

$$D_f(P_{data}||P_G) = \max_{\mathrm{D}} \{E_{x\sim P_{data}}[D(x)] - E_{x\sim P_G}\big[f^*\big(D(x)\big)\big]\}$$

$$G^* = arg\min_G D_f(P_{data}||P_G)$$

Original GAN has different V(G,D)

$$= arg\min_G \max_D \{E_{x\sim P_{data}}[D(x)] - E_{x\sim P_G}\big[f^*\big(D(x)\big)\big]\}$$

$$= arg\min_G \max_D V(G,D)$$ familiar? ☺

$$D_f(P_{data}||P_G) = \max_{D}\{E_{x\sim P_{data}}[D(x)] - E_{x\sim P_G}[f^*(D(x))]\}$$

Name	$D_f(P\|Q)$	Generator $f(u)$
Total variation	$\frac{1}{2}\int \lvert p(x) - q(x)\rvert \, \mathrm{d}x$	$\frac{1}{2}\lvert u-1\rvert$
Kullback-Leibler	$\int p(x) \log \frac{p(x)}{q(x)} \, \mathrm{d}x$	$u \log u$
Reverse Kullback-Leibler	$\int q(x) \log \frac{q(x)}{p(x)} \, \mathrm{d}x$	$-\log u$
Pearson χ^2	$\int \frac{(q(x)-p(x))^2}{p(x)} \, \mathrm{d}x$	$(u-1)^2$
Neyman χ^2	$\int \frac{(p(x)-q(x))^2}{q(x)} \, \mathrm{d}x$	$\frac{(1-u)^2}{u}$
Squared Hellinger	$\int \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 \, \mathrm{d}x$	$(\sqrt{u}-1)^2$
Jeffrey	$\int (p(x) - q(x)) \log \left(\frac{p(x)}{q(x)}\right) \, \mathrm{d}x$	$(u-1)\log u$
Jensen-Shannon	$\frac{1}{2}\int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} \, \mathrm{d}x$	$-(u+1)\log\frac{1+u}{2} + u \log u$
Jensen-Shannon-weighted	$\int p(x)\pi \log \frac{p(x)}{\pi p(x)+(1-\pi)q(x)} + (1-\pi)q(x) \log \frac{q(x)}{\pi p(x)+(1-\pi)q(x)} \, \mathrm{d}x$	$\pi u \log u - (1-\pi+\pi u)\log(1-\pi+\pi u)$
GAN	$\int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} \, \mathrm{d}x - \log(4)$	$u \log u - (u+1)\log(u+1)$

Name	Conjugate $f^*(t)$
Total variation	t
Kullback-Leibler (KL)	$\exp(t-1)$
Reverse KL	$-1 - \log(-t)$
Pearson χ^2	$\frac{1}{4}t^2 + t$
Neyman χ^2	$2 - 2\sqrt{1-t}$
Squared Hellinger	$\frac{t}{1-t}$
Jeffrey	$W(e^{1-t}) + \frac{1}{W(e^{1-t})} + t - 2$
Jensen-Shannon	$-\log(2 - \exp(t))$
Jensen-Shannon-weighted	$(1-\pi)\log\frac{1-\pi}{1-\pi e^{t/\pi}}$
GAN	$-\log(1-\exp(t))$

Using the f-divergence you like ☺

https://arxiv.org/pdf/1606.00709.pdf